Homework 8 (Stats 620, Winter 2017)

Due Thursday March 30, in class
Questions are derived from problems in *Stochastic Processes* by S. Ross.

1. Let $\{N(t)\}$ be a Poisson process with rate λ. Since $\{N(t)\}$ is also a continuous time Markov chain, it can also be defined in terms of its transition rates q_{ij} and rates ν_i of leaving i (together with its initial distribution $P_i(0) = \mathbb{P}\{N(0) = i\}$). For some fixed time T, let $N^*(t) = N(T-t)$ for $0 \le t \le T$. $\{N^*(t)\}$ is an inhomogeneous continuous time Markov chain, so it is specified by time-dependent parameters $q^*_{ij}(t)$ and $\nu^*_i(t)$ (together with its initial distribution $P^*_i(0) = \mathbb{P}\{N^*(0) = i\}$).

   (a) Write down expressions for q_{ij}, ν_i and $P_i(0)$.

   (b) Obtain expressions for $q^*_{ij}(t)$, $\nu^*_i(t)$ and $P^*_i(0)$. Note that q^*_{ij} does not depend on λ. Note also that duration in state i viewed in reverse time is different from forward in time (as mentioned in the notes, they are the same if the Markov chain is stationary).

   Hint: One approach is to employ Theorem 2.3.1 of Ross, which you may use without proof.

   Solution: (a) $\{N(t)\}$ has states 0, 1, 2,$\cdots$. The renewal process starts at 0, so $P_0(0) = 1$ and $P_i(0) = 0$ for $i > 0$. From the construction of Poisson processes, we have

$$P[N(t+h) = j | N(t) = i] = \begin{cases} \lambda h + o(h) & \text{for} \quad j = i+1 \\ o(h) & \text{for} \quad j > i+1, \\ 0 & \text{for} \quad j < i. \end{cases}$$

   Hence,

$$q_{ij} = \begin{cases} \lambda & \text{for } j = i+1, \\ 0 & \text{for } j > i+1 \text{ or } j < i. \end{cases}$$

   Since the only nonzero transition rate is $q_{i,i+1} = \lambda$, we have $\nu_i = \lambda$ (and in the Q matrix notation, $q_{i,i} = -\lambda$) for all $i = 0, 1, 2, \cdots$.

   (b) We apply Bayes' rule to get backward transition rates.

$$\begin{aligned} &P[N^*(t+h) = i-1 | N^*(t) = i] = P[N(T-t-h) = i-1 | N(T-t) = i] \\ &= \frac{P[N(T-t-h) = i-1] \cdot P[N(T-t) = i | N(T-t-h) = i-1]}{P[N(T-t) = i]}. \end{aligned}$$

   The above probability can be computed as

$$\frac{\frac{e^{-\lambda(T-t-h)}(\lambda(T-t-h))^{i-1}}{(i-1)!} \cdot (\lambda h + o(h))}{\frac{e^{-\lambda(T-t)}(\lambda(T-t))^{i}}{i!}} = \frac{i}{T-t}(h + o(h)).$$

   Thus we have $q^*_{i,i-1} = \frac{i}{T-t}$. The infinitesimal probabilities

$$P[N(T-t) = i | N(T-t-h) = j] = \begin{cases} o(h) & \text{for} \quad j < i-1 \\ 0 & \text{for} \quad j > i \end{cases}$$

similarly gives that $q^*_{i,j} = 0$ for $j > i$ or $j < i-1$. Therefore, $\lambda^*_i = -\frac{i}{T-t}$. The initial condition of the reverse chain is simply given by the Poisson distribution with rate λT.

$$P^*_i(0) = P_i(T) = \frac{e^{-\lambda T}(\lambda T)^i}{i!}.$$

2. The following problems arises in molecular biology. The surface of a bacterium consists of several sites at which foreign molecules—some acceptable and some not—become attached. We consider a particular site and assume that molecules arrive at the site according to a Poisson process with parameter λ. Among these molecules a proportion α are acceptable. Unacceptable molecules stay at the site for a length of time which is exponentially distributed with parameter μ_1, whereas an acceptable molecule remains at the site for an exponential time with departure rate μ_2. An arriving molecule will become attached only if the site is free of other molecules.

   (i) What percentage of the time is the site occupied with an acceptable molecule?

   (ii) What fraction of arriving acceptable molecules become attached?

   Solution:

   Consider a continuous–time Markov chain with 3 states. Define states 0, 1 and 2 as the site being free, attached to an unacceptable molecule and attached to an acceptable molecule respectively. Thus, the transition rate matrix Q is

$$Q = \begin{pmatrix} -\lambda & \lambda\alpha & \lambda(1-\alpha) \\ \mu_2 & -\mu_2 & 0 \\ \mu_1 & 0 & -\mu_1 \end{pmatrix}.$$

   (i) We have the balance equations

$$\begin{aligned} \mu_2 P_2 &= \alpha\lambda P_0 && (1)\\ \mu_1 P_1 &= (1-\alpha)\lambda P_0 && (2)\\ P_0 + P_1 + P_2 &= 1 && (3) \end{aligned}$$

   The long run fraction of time that an acceptable molecule is attached equals

$$P_1 = \frac{\alpha\mu_2^{-1}}{\lambda^{-1} + (1-\alpha)\mu_1^{-1} + \alpha\mu_2^{-1}}.$$

   (ii) Using the PASTA property (Poisson Arrivals See Time Averages), the fraction of arriving acceptable molecules that become attached equals the proportion that the site is free, which equals

$$P_0 = \frac{\lambda^{-1}}{\lambda^{-1} + (1-\alpha)\mu_1^{-1} + \alpha\mu_2^{-1}}.$$

3. An undirected graph has n vertices and edges between all $n(n-1)/2$ vertex pairs. A particle moves along the graph as follows: Events occur along the edge (i,j) according to independent Poisson processes with rates λ_{ij}. An event on edge (i,j) causes the edge to become "excited". If the particle is at vertex i the moment that the edge (i,j) becomes excited then the particle instantaneously moves from to vertex j. Let P_j denote the (limiting) proportion of the time that the particle is at vertex j. Explain why the position of the particle follows a continuous time Markov chain, and hence show that $P_j = 1/n$.

   Hint: use time reversibility.

   Solution:

   Let $X(t)$ denote the position of the particle at time t. Using Definition 2.1.2 of a Poisson process for $j \neq i$

$$\begin{aligned} & \mathbb{P}(X(t+h) = j | X(t) = i, X(s); s < t) \\ = \; & \mathbb{P}(\text{an event only on edge } (i,j) \text{ during } (t, t+h]) + o(h) \\ = \; & h\lambda_{ij} + o(h) \end{aligned}$$

   Thus $\{X(t), t \geq 0\}$ is a continuous–time Markov chain.
   Now $q_{ij} = \lambda_{ij} = \lambda_{ji} = q_{ji}$, since the graph is undirected. It is easy to verify that $P_j = 1/n$ satisfies $\sum_{j=1}^{n} P_j = 1$ and $P_i q_{ij} = P_j q_{ji}$ for $j \neq i$. Thus $\{X(t)\}$ is reversible with stationary distribution $\{P_j\}$. Also P_j is the proportion of the time that the particle is at vertex j.

4. Verify that $X_n/m^n, n \geq 1$, is a martingale when X_n is the size of the n^{th} generation of a branching process whose mean umber of offspring per individual is m.

   Solution:

   To check a sequence of random variables $Z_1, Z_2, \ldots$ is a martingale, we need to check

   (1) $\mathbb{E}|Z_n| < \infty \,, \forall n \in \mathbb{N}$;
   (2) $\mathbb{E}(Z_{n+1}|Z_1, \ldots, Z_n) = Z_n \,, \forall n \in \mathbb{N}$.

   In this case, we have

$$\mathbb{E}[Z_{n+1}|Z_1, \cdots, Z_n] = \mathbb{E}[Z_{n+1}|X_1, \cdots, X_n] = \mathbb{E}[\frac{X_{n+1}}{m^{n+1}}|X_1, \cdots, X_n] = \frac{1}{m^{n+1}} m X_n = Z_n \,,$$

   and

$$\mathbb{E}|Z_n| = \mathbb{E}Z_n = \mathbb{E}Z_1 = 1 \,.$$

   We have thus shown that Z_n is a martingale.

5. Consider the Markov chain which at each transition either goes up 1 with probability p or down 1 with probability $q = 1-p$. Argue that $(q/p)^{S_n}$, $n \geq 1$, is a martingale.

   Solution:

   Let $S_n = \sum_{i=1}^{n} X_i$. We have

$$\mathbb{E}|Z_n| = \mathbb{E}Z_n \leq \sum_{k=-n}^{n} (q/p)^k < \infty \,, \forall n \in \mathbb{N}$$

and

$$
\begin{aligned}
\mathbb{E}[Z_{n+1}|Z_1,\cdots,Z_n] &= \mathbb{E}[Z_{n+1}|S_1,\cdots,S_n] = \mathbb{E}[(q/p)^{X_{n+1}+S_n}||S_1,\cdots,S_n] \\
&= (q/p)^{S_n}\,\mathbb{E}[(q/p)^{X_{n+1}}|S_1,\cdots,S_n] = (q/p)^{S_n}\,\mathbb{E}[(q/p)^{X_{n+1}}] \\
&= Z_n\left((q/p)\,p + (q/p)^{-1}\,q\right) = Z_n\,.
\end{aligned}
$$

We have thus shown that Z_n is a martingale.

6. Consider a Markov chain $\{X_n, n \geq 0\}$ with $P_{NN} = 1$. Let $P(i)$ denote the probability that this chain eventually enters state N given that it starts in state i. Show that $\{P(X_n), n \geq 0\}$ is a martingale.

   Hint: One approach involves defining $A = \{X_\infty = N\}$ and showing that $\mathbb{E}\big[\mathbb{P}(A \mid X_n) \mid X_{n-1}] = \mathbb{E}\big[\mathbb{P}(A \mid X_n, X_{n-1}) \mid X_{n-1}\big] = \mathbb{P}(A \mid X_{n-1})$.

   <u>Solution</u>:

   First note that if $Z = g(Y)$ and $\mathbb{E}[X|Y] = h(g(Y))$ then

$$
\mathbb{E}[X|Z] = \mathbb{E}[\mathbb{E}[X|Y,Z]|Z] = \mathbb{E}[\mathbb{E}[X|Y]|Z] = \mathbb{E}[h(g(Y))|Z] = \mathbb{E}[h(Z)|Z] = h(Z)\,. \tag{4}
$$

   Now let $A = \{X_\infty = N\}$.

$$
\begin{aligned}
\mathbb{E}[\mathbb{P}(A|X_{n+1})|X_n,\cdots,X_1] &= \mathbb{E}[\mathbb{P}(A|X_{n+1})|X_n] \text{ (by Markov Property)} \\
&= \mathbb{E}[\mathbb{P}(A|X_{n+1},X_n)|X_n] \text{ (by Markov Property)} \\
&= \mathbb{P}(A|X_n) \text{ (by equation 6.1.2 Ross)}\,.
\end{aligned}
$$

   Thus, by (4),

$$
\mathbb{E}[\mathbb{P}(A|X_{n+1})|\mathbb{P}(A|X_n),\cdots,\mathbb{P}(A|X_1)] = \mathbb{P}(A|X_n)\,.
$$

   It is also clear that $\mathbb{E}(|\mathbb{P}(A|X_n)|) \leq 1$. Hence $\mathbb{P}(A|X_n)$ is a martingale.

Recommended reading:
Sections 5.6 (up to Prop. 5.6.3, i.e. pp 257–261), 6.1, 6.2.

Supplementary exercise: 6.2, 6.7
Optional, but recommended. Do not turn in solutions.