Homework 9 (Stats 620, Winter 2017)

Due Thursday April 6, in class
Questions are derived from problems in *Stochastic Processes* by S. Ross.

1. Consider successive flips of a coin having probability p of landing heads. Use a martingale argument to compute the expected number of flips until the following sequences appear:

   (a) HHTTHHT

   (b) HTHTHTH

2. Consider a sequence of independent tosses of a coin and let $\mathbb{P}\{\text{head}\}$ be the probability of a head on any toss. Let A be the hypothesis that $\mathbb{P}\{\text{head}\} = a$ and let B be the hypothesis that $\mathbb{P}\{\text{head}\} = b$, for $0 < a, b < 1$. Let X_i denote the outcome of the ith toss and let

$$Z_n = \frac{\mathbb{P}\{X_1, ...X_n \mid A\}}{\mathbb{P}\{X_1, ...X_n \mid B\}}$$

   Show that if B is true, then:

   (a) Z_n is a martingale, and

   (b) $\lim_{n\to\infty} Z_n$ exists with probability 1.

   (c) If $b \neq a$, what is $\lim_{n\to\infty} Z_n$?

3. An ordinary deck of cards is randomly shuffled and then the cards are exposed one at a time. At some time before all the cards have been exposed you must say "next", and if the next card exposed is a spade then you win and if not then you lose. For any strategy, show that at the moment you call "next" the conditional probability that you win is equal to the conditional probability that the last card is spade. Conclude from this that the probability of winning is 1/4 for all strategies.

   Hint: one approach is to show that the proportion of spades remaining is a martingale.

4. Let $\{S_n, n \geq 0\}$ denote a random walk in which X_i has distribution F. Let $G(t, s)$ denote the probability that the first value of S_n that exceeds t is less than or equal to $t + s$. That is,

$$G(t, s) = \mathbb{P}\{\text{first sum exceeding } t \text{ is } \leq t + s\}$$

   Show that

$$G(t, s) = F(t + s) - F(t) + \int_{-\infty}^{t} G(t - y, s) dF(y).$$

   Comment: The quantity $G(t, s)$ is interesting because it gives the distribution of the "overshoot" that is necessary to do a more accurate approximation than Equation 7.3.2 using the identity in Equation 7.3.1.

5. Let $X(t)$ be a standard Brownian motion and define $Y(t) = tX(1/t)$.

   (a) What is the distribution of $Y(t)$?

   (b) Compute $\mathrm{Cov}((Y(s), Y(t))$.

   (c) Argue that $\{Y(t), t \geq 0\}$ is also Brownian motion.

   (d) Let $T = \inf\{t > 0 : X(t) = 0\}$. Using (c) present an argument that $\mathbb{P}\{T = 0\} = 1$.

6. Let $X(t)$ be a standard Brownian motion and define $W(t) = X(a^2 t)/a$ for $a > 0$. Verify that $W(t)$ is also Brownian motion.

   Recommended reading:
   Sections 6.4, 7.3, 8.1, 8.2