Homework 9 (Stats 620, Winter 2017)

Due Thursday April 6, in class
Questions are derived from problems in *Stochastic Processes* by S. Ross.

1. Consider successive flips of a coin having probability p of landing heads. Use a martingale argument to compute the expected number of flips until the following sequences appear:

   (a) HHTTHHT

   (b) HTHTHTH

   Solution:

   Proceed as in Example 6.2 A (Ross). Let N denote the number of flips until the pattern appears. Imagine a sequence of gamblers, each initially having 1 unit playing at a fair gambling casino. Gambler i begins betting at the beginning of day i. At the beginning of each day another gambler starts playing. Let X_n be the total winnings of the casino after the nth day. Since all bets are fair, $\{X_n\}$ is a martingale with mean 0. At the end of day N, each of the gamblers $1, \cdots, N-7$ would have lost 1 unit. For gamblers $N-6, \cdots, N$ the observed pattern is a part of the stopping pattern.

   (a)

H	H	T	T	H	H	T
won $p^{-4}q^{-3}-1$	lost 1	lost 1	lost 1	won $p^{-2}q^{-1}-1$	lost 1	lost 1

   Hence
$$X_N = N - 2 - (p^{-4}q^{-3} - 1) - (p^{-2}q^{-1} - 1) = N - p^{-4}q^{-3} - p^{-2}q^{-1}\,.$$
   It is easy to verify condition (iii) of the Martingale Stopping Theorem, 6.2.2 (Ross). Thus $\mathbb{E}[X_N] = 0$ or
$$\mathbb{E}[N] = p^{-4}q^{-3} + p^{-2}q^{-1}\,.$$

   (b)

H	T	H	T	H	T	H
won $p^{-4}q^{-3}-1$	lost 1	won $p^{-3}\text{–}2-1$	lost 1	won $p^{-2}q^{-1}-1$	lost 1	won $p^{-1}-1$

   Hence
$$X_N = N - p^{-4}q^{-3} - p^{-3}\text{–}2 - p^{-2}q^{-1} - p^{-1}\,.$$
   It is easy to verify condition (iii) of the Martingale Stopping Theorem, 6.2.2 (Ross). Thus $\mathbb{E}[X_N] = 0$ or
$$\mathbb{E}[N] = p^{-4}q^{-3} + p^{-3}q^{-2} + p^{-2}q^{-1} + p^{-1}\,.$$

2. Consider a sequence of independent tosses of a coin and let $\mathbb{P}\{\text{head}\}$ be the probability of a head on any toss. Let A be the hypothesis that $\mathbb{P}\{\text{head}\} = a$ and let B be the hypothesis that $\mathbb{P}\{\text{head}\} = b$, for $0 < a, b < 1$. Let X_i denote the outcome of the ith toss and let
$$Z_n = \frac{\mathbb{P}\{X_1, ...X_n \mid A\}}{\mathbb{P}\{X_1, ...X_n \mid B\}}$$

Show that if B is true, then:

(a) Z_n is a martingale, and

(b) $\lim_{n\to\infty} Z_n$ exists with probability 1.

(c) If $b \neq a$, what is $\lim_{n\to\infty} Z_n$?

Solution:

(a) Let $X_i = 1$ if the outcome of the ith toss is head, $X_i = 0$ if it is tail. From independence,

$$Z_n = \frac{P(X_1, \cdots, X_n|A)}{P(X_1, \cdots, X_n|B)} = Z_{n-1}\frac{P(X_n|A)}{P(X_n|B)} \tag{1}$$

Now

$$\mathbb{E}\left[\frac{P(X_n|A)}{P(X_n|B)}\right] = \mathbb{E}\left[\frac{a^{X_n}(1-a)^{1-X_n}}{b^{X_n}(1-b)^{1-X_n}}\right] = \frac{a}{b}b + \frac{1-a}{1-b}(1-b) = 1\,.$$

Further, as shown in the following $E|Z_n| < \infty$. Thus $\{Z_n\}$ is a product martingale.

(b) $Z_n > 0$. Thus $E|Z_n| = \mathbb{E}[Z_n]$. Since $\{Z_n\}$ is a product martingale $\mathbb{E}[Z_n] = 1$ and hence bounded. So by Martingale Convergence Theorem $\lim_{n\to\infty} Z_n$ exists and is finite with probability 1.

(c) From equation (1) it is clear that Z_n can have a finite (random or constant) non- zero limit only if

$$\lim_{n\to\infty} \frac{P(X_n|A)}{P(X_n|B)} = 1$$

with probability 1. However for $a \neq b$ it is not possible. Thus the limit is 0.

Another approach. Let $S_n = \sum_{i=1}^n X_i$. Note that

$$Z_n = \prod_{i=1}^n \frac{a^{X_i}(1-a)^{1-X_i}}{b^{X_i}(1-b)^{1-X_i}} = \left(\frac{a}{b}\right)^{S_n}\left(\frac{1-a}{1-b}\right)^{n-S_n} = \left(\frac{a(1-b)}{b(1-a)}\right)^{S_n}\left(\frac{1-a}{1-b}\right)^{n},$$

and thus with probability 1,

$$\frac{\log Z_n}{n} = \frac{S_n}{n}\log\left(\frac{a(1-b)}{b(1-a)}\right) + \log\left(\frac{1-a}{1-b}\right) \longrightarrow b\log\left(\frac{a(1-b)}{b(1-a)}\right) + \log\left(\frac{1-a}{1-b}\right) \quad \text{as } n\to\infty$$

Note that

$$\begin{aligned} b\log\left(\frac{a(1-b)}{b(1-a)}\right) + \log\left(\frac{1-a}{1-b}\right) &= b\log\left(\tfrac{a}{b}\right) + (1-b)\log\left(\tfrac{1-a}{1-b}\right) \\ &\leq \log\left(b\tfrac{a}{b} + (1-b)\tfrac{1-a}{1-b}\right) = 0\,, \end{aligned}$$

where the last inequality follows from Jensen's inequality, and the equality holds only when $\frac{a}{b} = \frac{1-a}{1-b}$, i.e., $a = b$. Hence, we have shown that

$$\lim_{n\to\infty} \frac{\log Z_n}{n} < 0 \quad \text{a.e.}\,,$$

which implies that $\lim_{n\to\infty} Z_n = 0 \quad$ a.e..

3. An ordinary deck of cards is randomly shuffled and then the cards are exposed one at a time. At some time before all the cards have been exposed you must say "next", and if the next card exposed is a spade then you win and if not then you lose. For any strategy, show that at the moment you call "next" the conditional probability that you win is equal to the conditional probability that the last card is spade. Conclude from this that the probability of winning is 1/4 for all strategies.

   Hint: one approach is to show that the proportion of spades remaining is a martingale.

   Solution:

   Let X_n indicate if the nth card is a spade and Z_n be the proportion of spades in the remaining cards after the n card. Thus $E|Z_n| < \infty$ and

$$\mathbb{E}[Z_n|Z_{n-1},\cdots,Z_1] = \frac{(52-n+1)Z_{n-1}-1}{52-n+1}Z_{n-1} + \frac{(52-n+2)Z_{n-1}}{52-n+1}(1-Z_{n-1}) = Z_{n-1}\,.$$

   Hence $\{Z_n\}$ is a martingale.
   Note that $X_{52} = Z_{51}$. Thus

$$\begin{aligned}\mathbb{E}[X_{n+1}|X_n,\cdots,X_1] &= \mathbb{E}[X_{n+1}|Z_n,\cdots,Z_1] = Z_n \\ &= \mathbb{E}[Z_{51}|Z_n,\cdots,Z_1] = \mathbb{E}[X_{52}|X_n,\cdots,X_1]\,.\end{aligned}$$

   Finally, let N be the stopping time corresponding to saying "next" for a given strategy.

$$\begin{aligned}P(\text{Win}) &= \mathbb{E}[X_{N+1}] = \mathbb{E}[\mathbb{E}[X_{N+1}|N]] \\ &= \mathbb{E}[Z_N] = \mathbb{E}[Z_1] = 1/4\,. \qquad (2)\end{aligned}$$

   where equation (2) uses the Martingale Stopping Theorem.

4. Let $\{S_n, n \geq 0\}$ denote a random walk in which X_i has distribution F. Let $G(t,s)$ denote the probability that the first value of S_n that exceeds t is less than or equal to $t+s$. That is,

$$G(t,s) = \mathbb{P}\{\text{first sum exceeding } t \text{ is } \leq t+s\}$$

   Show that

$$G(t,s) = F(t+s) - F(t) + \int_{-\infty}^{t} G(t-y,s)dF(y).$$

   Comment: The quantity $G(t,s)$ is interesting because it gives the distribution of the "overshoot" that is necessary to do a more accurate approximation than Equation 7.3.2 using the identity in Equation 7.3.1.

   Solution:

   $S_n|X_1$ is distributed as $X_1 + S_{n-1}$. Thus if $A = \{\text{first sum exceeding } t \text{ is } \leq t+s\}$,

$$\begin{aligned}G(t,s) &\equiv P(A) = \mathbb{E}[P(A|X_1)] \\ &= F(t+s) - F(t) + \int_{-\infty}^{t} G(t-y,s)\mathrm{d}F(y)\,.\end{aligned}$$

5. Let $X(t)$ be a standard Brownian motion and define $Y(t) = tX(1/t)$.

   (a) What is the distribution of $Y(t)$?

   (b) Compute $\mathrm{Cov}((Y(s), Y(t))$.

   (c) Argue that $\{Y(t), t \geq 0\}$ is also Brownian motion.

   (d) Let $T = \inf\{t > 0 : X(t) = 0\}$. Using (c) present an argument that $\mathbb{P}\{T = 0\} = 1$.

   Solution:

   (a) The characteristic function of $Y(t)$

$$\mathbb{E}[\exp(isY(t))] = \mathbb{E}[\exp(istX(1/t))] = \exp\left(\frac{-s^2t^2}{2t}\right) = \exp\left(\frac{-s^2t}{2}\right).$$

   Thus $Y(t) \sim N(0, t)$.

   (b)

$$\begin{aligned}\mathrm{Cov}(Y(s), Y(t)) &= \mathrm{Cov}(sX(1/s), tX(1/t)) \\ &= st\ \mathrm{Cov}(X(1/s), X(1/t)) = st\ \min(1/s, 1/t) \\ &= \min(s, t).\end{aligned}$$

   (c) Since $\{X(t)\}$ is a Gaussian process so is $\{Y(t)\}$. Further from parts (a) and (b) above $\{Y(t)\}$ is a Brownian Motion.

   (d) Since $Y(t)$ is Brownian Motion then $T_1 \equiv \sup\{t : Y(t) = 0\} = \infty$ with probability 1. Note $\{T = 0\} = \{T_1 = \infty\}$. Thus $P(T = 0) = 1$.

6. Let $X(t)$ be a standard Brownian motion and define $W(t) = X(a^2t)/a$ for $a > 0$. Verify that $W(t)$ is also Brownian motion.

   Solution: $W(0) = X(0)/a = 0$. Non-overlapping increments of $W(t)$ map to non-overlapping increments of $X(t)$. Thus increments of $W(t)$ are independent. Further, for $s < t$,

$$W(t) - W(s) = \frac{X(a^2t) - X(a^2s)}{a} \stackrel{D}{=} X(a^2(t-s))/a \sim N(0, t-s).$$

   Thus $W(t)$ has stationary increments with required distribution. Therefore $W(t)$ is a Brownian Motion.

   Recommended reading:
   Sections 6.4, 7.3, 8.1, 8.2