2. The Poisson Process

A counting process $\{N(t), t \geq 0\}$ is a **Poisson process** with rate λ if $\ldots$

<u>Definition 1.</u>

(i) $N(0) = 0$,
(ii) $N(t)$ has independent increments,
(iii) $N(t) - N(s) \sim \text{Poisson}\ (\lambda(t-s))$ for $s < t$.

This can be shown to be equivalent to

<u>Definition 2.</u>

(i) $N(0) = 0$,
(ii) $N(t)$ has stationary independent increments,
(iii) $\mathbb{P}\left(N(h) = 1\right) = \lambda h + o(h)$,
(iv) $\mathbb{P}\left(N(h) \geq 2\right) = o(h)$.

$$\boxed{f(h) = o(h) \text{ means } \lim_{h \to 0} \frac{f(h)}{h} = 0}$$

- Definition 1 is good for explicit calculations with Poisson processes. Definition 2 is useful for showing whether a process of interest is a Poisson process.

To Show Definition 2 Implies Definition 1

- We need only to show that Definition 2 implies $N(t) \sim \text{Poisson}\,(\lambda t)$. We divide $[0,t]$ into n equal subintervals and define
$$X_{nk} = \begin{cases} 1 & \text{if } N\big(k\,t/n\big) - N\big((k-1)\,t/n\big) \geq 1 \\ 0 & \text{else} \end{cases}$$
Then set $X_n = \sum_{k=1}^{n} X_{nk}$, so X_n counts the number of subintervals with at least one event.

- We aim to show that
(a) $\lim_{n\to\infty} \mathbb{P}\,[X_n = N(t)] = 1$, i.e. for sufficiently large n there is only ever 0 or 1 event per subinterval.
(b) $\lim_{n\to\infty} \mathbb{P}\,[X_n = k] = (\lambda t)^k e^{-\lambda t}/k!$

- (a) and (b) together imply $N(t) \sim \text{Poisson}\,(\lambda t)$.

Note: Ross uses a rather different proof—see Theorem 2.1.1.

Proof

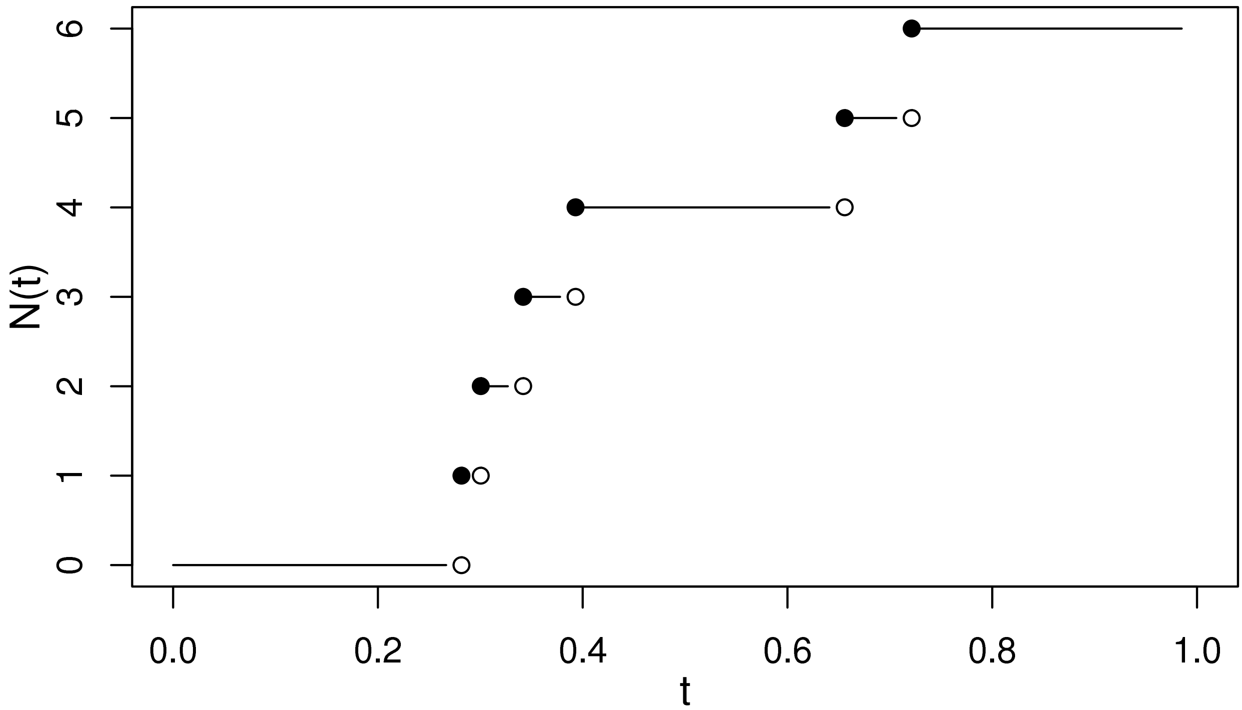
Proof continued

Interarrival and Waiting Times

- Let $X_1, X_2, \ldots$ be iid $\text{Exponential}\,(\lambda)$ variables. Define $S_n = \sum_{i=1}^{n} X_i$ and $S_0 = 0$. Set $N(t) = \max\{n : S_n \leq t\}$. Then $N(t)$ is a counting process which increases by one at times $S_1, S_2, \ldots$

- S_n is the n^{th} **arrival time**, or the **waiting time** until the n^{th} event. $X_1, X_2, \ldots$ are the **interarrival times**.

Definition 3. $N(t)$ constructed as above is a Poisson process of rate λ.



- Note that $\{N(t) \geq n\} = \{S_n \leq t\}$. Why?

Which of the following are true?

(a) $\{N(t) < n\} = \{S_n > t\}$

(b) $\{N(t) \leq n\} = \{S_n \geq t\}$

(c) $\{N(t) > n\} = \{S_n < t\}$

Definition 1 $\Rightarrow$ Definition 3

(i) Show that $\mathbb{P}[X_1 > t] = e^{-\lambda t}$.

(ii) Show that $\mathbb{P}[X_2 > t | X_1 = s] = e^{-\lambda t}$.

(iii) Repeating the same argument inductively gives Definition 3.

Definition 3 $\Rightarrow$ Definition 2

- The tricky thing is to show that Definition 3 implies stationary, independent increments. This follows from the memoryless property....

proof

proof continued

Example (Conditional Arrival Times)

If $N(t)$ is a rate λ Poisson process then, conditional on $N(t) = n$, the arrival times $S_1, \ldots, S_n$ have the same distribution as $U_{(1)}, \ldots, U_{(n)}$, the order statistics for $U_1, \ldots, U_n \sim \text{iid}\, U[0, t]$.

- From a previous example, this implies that, conditional on $N(t) = n$, $S_k = tB_k$ where $B_k \sim \text{Beta}\,(k, n-k+1)$.
- Conditional on $N(t) = 1$, $S_1 \sim U[0, t]$.
- The rate λ is seen to be irrelevant once one conditions on $N(t) = n$.

Proof

proof continued

Example (A Reward Process)

Suppose events occur as a Poisson process, rate λ. Each event S_k leads to a reward X_k which is an independent draw from $F_s(x)$ conditional on $S_k{=}s$. The total reward at t is $R = \sum_{k=1}^{N(t)} X_k$. Show that R has the same distribution as $\sum_{k=1}^{N(t)} Y_k$ where $Y_1, Y_2, \ldots$ are suitable independent random variables and use this to find $\mathbb{E}[R]$ and $\mathrm{Var}(R)$.

- Note: $X_1, X_2, \ldots$ are **not** independent, since $S_1, S_2, \ldots$ are not independent. They are conditionally independent, given $S_1, S_2, \ldots$ (what does this mean?)

Solution

Solution continued

Examples of Counting Processes

- **Queues**. Individuals arrive randomly and wait for service. E.g., customers at a store, parts in an assembly line, telephone calls at an exchange, cellular telephone calls at an antenna, internet packets at a router.

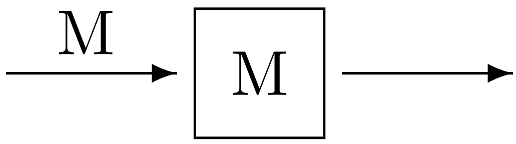
- **Networks**. Individuals move round a network of connected notes. Counting processes keep track of their locations. Each mode may be modeled by a queue. E.g., internet networks, social networks—individuals infected with a disease move around, contacting friends and colleagues.

- **Populations** Birth/death processes and predator/prey models, e.g. survival of endangered species.

- **Genetics**. Mutations arise randomly during reproduction. Thus harmless mutations may occur as a Poisson process (with "time" being length along the genome). For disease inheritance, cross-over events when parental chromosomes are combined during reproduction are important—these also occur as (approximately) a Poisson process along the genome.

Some Terminology for Queues

- **M/M/1** queue:



Exponential arrival intervals (M for Markov), i.e., arrivals are a Poisson process. Service times are independent and Exponential. Arrivals wait until the server is available, and they are served in order of arrival. What is the mean waiting time? The distribution of wait times?

- **M/G/1** queue: Markov arrivals, general service time, 1 server.
- **G/M/1** queue: General inter-arrival distribution, Exponential Service times, 1 server.
- **M/G/∞** Markov arrivals are immediately attended by a server, with general service time distribution.

Nonhomogeneous Poisson Processes

A counting process $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process with rate $\lambda(t)$ if:

Definition 1.

(i) $N(0) = 0$.

(ii) $N(t)$ has independent increments.

(iii) $N(t) - N(s) \sim \text{Poisson}\left(\int_s^t \lambda(u)\,du\right)$.

Definition 2.

(i) $N(0) = 0$.

(ii) $N(t)$ has independent increments.

(iii) $\mathbb{P}\left[N(t+h) - N(t) = 1\right] = h\lambda(t) + o(h)$

(iv) $\mathbb{P}\left[N(t+h) - N(t) \geq 2\right] = o(h)$

Definition 3. Setting $S_n = \sum_{k=1}^n X_k$ to define the inter-arrival and arrival times, X_{n+1} is conditionally independent of $X_1, \ldots, X_n$ given S_n, and has a distribution given by

$\mathbb{P}\left[X_{n+1} > t | S_n = s\right] = \exp\left\{-\int_s^{s+t} \lambda(u)\,du\right\}$.

Example (Splitting a Poisson Process)

Let $\{N(t)\}$ be a Poisson process, rate λ. Suppose that each event is randomly assigned into one of two classes, with time-varing probabilities $p_1(t)$ and $p_2(t)$. Each assignment is independent. Let $\{N_1(t)\}$ and $\{N_2(t)\}$ be the counting process for events of each class. Then $\{N_1(t)\}$ and $\{N_2(t)\}$ are independent nonhomogenous Poisson processes with rates $\lambda p_1(t)$ and $\lambda p_2(t)$.

- note: the independence is surprising, since the assignment of events appears to introduce dependence.

(i) show that $\{N_1(t)\}$ and $\{N_2(t)\}$ satisfy Definition 2, with the required rates.

(ii) Use a partitioning method to argue that $\{N_1(t)\}$ and $\{N_2(t)\}$ have the same joint probabilities as two independent Poisson processes.

Example (Rescaling Time)

For $\{N(t)\}$ a nonhomogeneous Poisson process with rate $\lambda(t)$, set $m(t) = \int_0^t \lambda(s)\, ds$ and define $N^*(t) = N(m^{-1}(t))$. Show that $\{N^*(t)\}$ is a Poisson process with rate 1.