Applied Probability Qualifying Review Exam 2008: Solutions

1. *Four children are playing two video games. The first game, which takes an average of* 4 *minutes to play, is not very exciting, so when a child completes a turn on it they always stand in line to play the other one. The second one, which takes an average of* 8 *minutes, is more interesting so, upon completing the game, the child will get back in line to play it with probability* 1/2 *or go to the other machine with probability* 1/2*. Assuming that they turns take an exponentially distributed amount of time, find the stationary distribution of the number of children playing or in line at each of the two machines.*

   Let $X(t)$ denote the number of children at game 1 at time t. $X(t)$ fully determines the state of the system, which can be represented by a Markov chain on $0, 1, 2, 3, 4$ with $q_{i,i+1} = 1/(16 \text{ min}) = \lambda$ for $i = 0, 1, 2, 3$ and $q_{i,i-1} = 1/(4 \text{ min}) = 4\lambda$ for $i = 1, 2, 3, 4$. Solving the detailed balance equations gives a stationary distribution of

$$P_i = (1/4)^i \Big/ \sum_{k=0}^{4} (1/4)^k.$$

2. *Let T_n be the time of the nth arrival in a Poisson process $N(t)$ with intensity λ, and define the excess lifetime process $L(t) = T_{N(t)+1} - t$, being the time one must wait subsequent to t before the next arrival. Show by conditioning on T_1 that*

$$P[L(t) > x] = e^{-\lambda(t+x)} + \int_0^t P[L(t-\mu) > x]\, \lambda e^{-\lambda\mu}\, d\mu.$$

   Solve this integral equation in order to find the distribution function of $L(t)$. Discuss the interpretation of your conclusion.

   Conditioning on T_1 and using the time-homogeneity of the process

$$P[L(t) > x | T_1 = u] = \begin{cases} P[L(t-u) > x] & \text{if } u \le t \\ 0 & \text{if } t < u \le t+x \\ 1 & \text{if } u > t+x \end{cases}$$

   (drawing a diagram helps to see this). Therefore

$$\begin{aligned} P[L(t) > x] &= \int_0^\infty P[L(t) > x | T_1 = u] \lambda e^{\lambda u} du \\ &= \int_0^t P[L(t-u) > x] \lambda e^{\lambda u} du + \int_{t+x}^\infty \lambda e^{-\lambda u} du. \end{aligned}$$

   One may solve the integral equation using Laplace transforms. Alternately you may guess the answer and then check that it works. The answer is $P[L(t) \le x] = 1 - e^{-\lambda x}$, the exponential distribution. This answer is obvious since $E(t) > x$ if and only if there is no arrive in $[t, t+x]$, an event having probability $e^{-\lambda x}$.

3. *Let $\{Y_n\}$ be a martingale with $E[Y_n] = 0$ and $E[Y_n^2] < \infty$ for all n. Show that*

$$P\left(\max_{1 \le k \le n} Y_k > x\right) \le \frac{E(Y_n^2)}{E(Y_n^2) + x^2},$$

for any $x > 0$.

Hint: If you wish you may use, without proof, Kolmogorov's submartingale inequality. Namely, if $\{Z_n\}$ *is a non-negative submartingale then*

$$P[\max(Z_1, \ldots, Z_n) > a] \leq \frac{E[Z_n]}{a}.$$

$$P\left(\max_{1 \leq k \leq n} Y_k > x\right) \leq P\left(\max_{1 \leq k \leq n} (Y_k + c)^2 > (x + c)^2\right). \tag{1}$$

Now $(Y_k + c)^2$ is a convex function of Y_k, and therefore defines a submartingale. Applying Kolmogorov's submartingale inequality to this, we obtain an upper bound of $E[(Y_n + c)^2]/(x + c)^2$ for the right-hand side of (1). We set $c = \mathbf{E}(Y_n^2)/x$ to obtain the result.

4. *Let* $B(t)$ *be a standard Brownian motion, and let* $F(\mu, \nu)$ *be the event that* $B(t)$ *has no zero in the interval* (μ, ν). *If* $ab > 0$, *show that*

$$P[\, F(0,t) \,|\, B(0) = a, B(t) = b \,] = 1 - e^{-2ab/t}.$$

We may assume that $a, b > 0$. With

$$p_t(b) = P[B(t) > b, F(0,t) | B(0) = a],$$

we have by duality (i.e., the reflection principle) that

$$\begin{aligned} p_t(b) &= P[B(t) > b | B(0) = a] - P[B(t) < -b | B(0) = a] \\ &= P[b - a < B(t) < b + a | B(0) = 0], \end{aligned}$$

giving that

$$\frac{\partial p_t(b)}{\partial b} = f(b + a, t) - f(b - a, t)$$

where $f(x,t)$ is the $N(0,t)$ density function. Now, using conditional probabilities,

$$P[F(0,t) | B(0) = a, B(t) = b] = -\frac{1}{f(b-a,t)} \frac{\partial p_t(b)}{\partial b} = 1 - e^{-2ab/t}.$$