Applied Probability Qualifying Review Exam Questions
May, 2008

Each question is worth 10 points. In the exam, full credit required solving 7 of 8 questions (these four plus four questions based on Stat 621) in four hours.

1. Four children are playing two video games. The first game, which takes an average of 4 minutes to play, is not very exciting, so when a child completes a turn on it they always stand in line to play the other one. The second one, which takes an average of 8 minutes, is more interesting so, upon completing the game, the child will get back in line to play it with probability 1/2 or go to the other machine with probability 1/2. Assuming that they turns take an exponentially distributed amount of time, find the stationary distribution of the number of children playing or in line at each of the two machines.

2. Let T_n be the time of the nth arrival in a Poisson process $N(t)$ with intensity λ, and define the excess lifetime process $L(t) = T_{N(t)+1} - t$, being the time one must wait subsequent to t before the next arrival. Show by conditioning on T_1 that

$$P[L(t) > x] = e^{-\lambda(t+x)} + \int_0^t P[L(t-\mu) > x]\, \lambda e^{-\lambda\mu}\, d\mu.$$

   Solve this integral equation in order to find the distribution function of $L(t)$. Discuss the interpretation of your conclusion.

3. Let $\{Y_n\}$ be a martingale with $E[Y_n] = 0$ and $E[Y_n^2] < \infty$ for all n. Show that

$$P\left(\max_{1\leq k\leq n} Y_k > x\right) \leq \frac{E(Y_n^2)}{E(Y_n^2) + x^2},$$

   for any $x > 0$.

   Hint: If you wish you may use, without proof, Kolmogorov's submartingale inequality. Namely, if $\{Z_n\}$ is a non-negative submartingale then

$$P[\max(Z_1, \ldots, Z_n) > a] \leq \frac{E[Z_n]}{a}.$$

4. Let $B(t)$ be a standard Brownian motion, and let $F(\mu, \nu)$ be the event that $B(t)$ has no zero in the interval (μ, ν). If $ab > 0$, show that

$$P[\, F(0,t) \,|\, B(0) = a, B(t) = b\,] = 1 - e^{-2ab/t}.$$