Applied Probability Qualifying Review Exam Questions
May, 2009
time allowed: approximately 1/2 hr per question

1. Consider a continuous-time Markov chain $\{X(t), t \geq 0\}$ with stationary probabilities $\{P_i, i \geq 0\}$. Denote the transition rates by $\{q_{ij}\}$ and the rate of leaving i by $\nu_i = \sum_{j \neq i} q_{ij}$. Let T denote the first time that the chain has been in state 0 for τ consecutive time units. Find $E[T \mid X(0) = 0]$.

2. Let $B(t)$ be a standard Brownian motion started at $b > 0$, i.e., $B(0) = b$ and $B(t) - B(s) \sim N\big(0, |t-s|\big)$. Let T be the first time that $B(t)$ hits zero, i.e., $T = \min\{t : B(t) = 0\}$. Brownian motion with absorption at zero is $Y(t) = B(t \wedge T)$, so $Y(t)$ follows the path of the Brownian motion until the first visit to zero and then stays there forever. Calculate the transition probablity density function $p(x, y, t)$ of $Y(t)$, defined such that

$$P\Big(y_0 < Y(s+t) < y_1 \mid Y_s = x\Big) = \int_{y_0}^{y_1} p(x, y, t)\, dy$$

for $0 < y_0 < y_1$.

3. This question studies a simple version of the Wright-Fisher model, which is fundamental to the study of population genetics. Consider a population containing N copies of a gene that can each be one of two types, A or a. (Since humans have two copies of each gene, this would correspond to the N copies of a gene belonging to $N/2$ individuals.) We model the number of genes ot type A in successive generations, supposing that each generation has the same fixed number N of copies of the gene. Specifically, let X_n be the number of genes of type A in the nth generation, so X_n takes values in $\{0, 1, \ldots, N\}$. Suppose a model of reproduction in which the population of genes at time $n+1$ is obtained by drawing N times with replacement from the population at time n. Thus, X_n is a Markov chain with transition probabilities given by

$$p(i,j) = \binom{N}{j}\Big(\frac{i}{N}\Big)^j\Big(1 - \frac{i}{N}\Big)^{N-j}.$$

(a) Show that $Y_n = X_n(N - X_n)/(1 - 1/N)^n$ is a martingale.

(b) Hence, show that

$$(N-1) \leq \frac{x(N-x)(1-1/N)^n}{P(0 < X_n < N \mid X(0) = x)} \leq \frac{N^2}{4}.$$

(This gives a bound on the chance that either type a or A has spread through the entire population by time n.)

4. Let S_n be a simple nearest-neighbor random walk on the integers (i.e., $P\{S_{n+1}{=}S_n + 1\} = P\{S_{n+1}{=}S_n - 1\} = 1/2$), started at $S_0 = 1$. Define T to be the time of the first visit to the origin, i.e., $T = \min\{n : S_n = 0\}$. Define $Z_0 = 1$ and

$$Z_k = \sum_{n=0}^{T-1} \mathbf{1}\{S_n = k \text{ and } S_{n+1} = k+1\}.$$

Thus, Z_k is the number of times that the random walk S_n crosses from k to $k+1$ before first visiting 0.

(a) Show that the sequence $\{Z_k, k \geq 0\}$ is a Galton-Watson branching process, and identify the offspring distribution as a geometric distribution.

Hint: recall that X_n is a Galton-Watson process if it can be written as $X_{n+1} = \sum_{m=1}^{X_n} C_{nm}$ where $\{C_{nm}\}$ is an independent, identically distributed collection of non-negative integer-valued random variables.

(b) The probability generating function of Z_k is $\phi_k(s) = \sum_{j=0}^{\infty} s^j P(Z_k = j)$. Show how to write $\phi_k(s)$ in terms of $\phi_1(s)$, and use part (a) to find an explicit expression for $\phi_1(s)$.