Applied Probability Qualifying Review Exam Questions
May, 2010
Time allowed: approximately 1/2 hr per question

1. Let $\{X_n, n \geq 0\}$ be a Markov chain taking values in $\{1, 2, \ldots, m\}$ and having one-step transition probabilities $P_{ij} = \mathbb{P}[X_n = j \mid X_{n-1} = i]$. Let $\mathcal{A}$ be a subset of the statespace, $\mathcal{A} \subset \{1, 2, \ldots, m\}$, and define $N = \min\{n : X_n \in \mathcal{A}\}$. Find a set of equations whose solution gives the following:

   (i) The mean time until $\{X_n\}$ hits $\mathcal{A}$ if the chain is started at i, i.e., $\mathbb{E}[N \mid X_0 = i]$.

   (ii) The probability that $\{X_n\}$ first hits $\mathcal{A}$ at j, i.e., $\mathbb{P}[X_N = j \mid X_0 = i]$.

   Solution: Let $m_i = \mathbb{E}[N \mid X_0 = i]$. Conditioning on X_1 gives

   $$m_i = 1 + \sum_{j \notin \mathcal{A}} P_{ij} m_j.$$

   Also, letting $\phi_{ij} = \mathbb{P}[X_N = j \mid X_0 = i]$, conditioning on X_1 gives

   $$\phi_{ij} = P_{ij} + \sum_{k \notin \mathcal{A}} P_{ik} \phi_{kj}.$$

2. This question considers a model for the fraction X_n of individuals in the nth generation of a certain population having a particular genetic trait (say, a gene for male-pattern baldness). Thinking of the population as being large, we suppose X_n is a continuous-valued random variable taking values in $[0, 1]$. In addition, we suppose that $\{X_n, n \geq 0\}$ has the Markov property. Further, we suppose that the genetic trait is neutral, meaning that it neither helps nor harms the individual, and so $\mathbb{E}[X_n \mid X_{n-1}] = X_{n-1}$. To allow for chance variation in transmission of the trait between generations, we suppose that $\mathrm{Var}[X_n \mid X_{n-1}] = cX_{n-1}(1 - X_{n-1})$ where c is some unknown constant. This can be thought of as generalizing the binomial distribution. Finally, suppose that $X_0 = p$ for some $0 < p < 1$.

   (i) Prove that X_n converges in some specified sense to either zero or one, i.e., in this model the trait will eventually die out or spread through the entire population.

   (ii) Find the probability that the trait eventually spreads through the entire population.

   (iii) Use a martingale stopping argument to find a lower bound on the probability that the fraction of the population with the trait never exceeds q for $q > p$.

   Solution: X_n is clearly a martingale. Since it is also non-negative, $\lim_{n\to\infty} X_n$ exists almost surely. But by construction sample paths cannot converge to any value in $(0, 1)$, so the limit points must be either 0 or 1. It follows that

   $$\mathbb{P}[\lim_{n\to\infty} X_n = 1] = \mathbb{E}[\lim_{n\to\infty} X_n] = \lim_{n\to\infty} \mathbb{E}[X_n] = p,$$

   with the interchange of limit and expectation justified for uniformly bounded random variables.

Now consider the stopping time $N = \min\{n : X_n \leq \epsilon \text{ or } X_n \geq q\}$. Let $A = \{X_N \geq q\}$ and $B = \{X_N \leq \epsilon\}$. The martingale stopping theorem applies as $\{X_n\}$ is bounded, and so

$$\begin{aligned} p &= \mathbb{E}[X_N|A]\mathbb{P}[A] + \mathbb{E}[X_N|B]\mathbb{P}[B] &(1)\\ &\geq q\mathbb{P}[A]. &(2) \end{aligned}$$

Thus, $\mathbb{P}[A] \leq p/q$. Taking $\epsilon \to 0$, we find $\mathbb{P}[\sup_k X_k \geq q] \leq p/q$ and so $\mathbb{P}[\sup_k X_k < q] \geq 1 - p/q$.

3. Let $N(t)$ be a counting process describing customers arriving at a queue, with W_n being the time spent in the queue by the nth customer and $X(t)$ being the number of customers in the queue at time t. Let $\{S_n, n \geq 1\}$ be the sequence of times at which $X(t)$ jumps from zero to one. Suppose that these are regeneration times for the system, i.e., $\{X(t+u), t \geq 0\}$ is conditionally independent of $\{X(s), 0 \leq s \leq u\}$ given $\{S_n = u\}$ and has a conditional distribution which does not depend on n. In other words, the system resets when the queue is empty. Suppose that $\mathbb{E}[S_n - S_{n-1}] = \mu < \infty$. Beyond these assumptions, we allow consideration of an arbitrary queueing structure. Show that

$$L = \lambda W$$

where $\quad L = \lim_{t\to\infty} \frac{1}{t}\int_0^t X(u)\,du, \quad \lambda = \lim_{t\to\infty} N(t)/t, \quad$ and $\quad W = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n W_k.$

Solution: This is a delayed regerative process, as $t = 0$ is not a regeneration time. However, we will modify it by placing a regeneration time at $t = 0$; this is notationally convenient and has no consequences for the limit results. Let N be the number of customers arriving (and therefore also exiting) the queue in the interval $[0, S_1)$. Via the renewal-reward theorem, the limits L, λ and W exist and are given by

$$L = \mathbb{E}\left(\int_0^{S_1} X(u)\,du\right) \Big/ \mathbb{E}[S_1], \quad \lambda = \mathbb{E}[N]/\mathbb{E}[S_1], \quad W = \mathbb{E}\left(\sum_{n=1}^N W_n\right) \Big/ \mathbb{E}[N]. \qquad (3)$$

Defining $I_n(t) = \begin{cases} 1 & \text{if the } n\text{th customer is in the system at time } t \\ 0 & \text{else} \end{cases}$,
we have $W_n = \int_0^{S_1} I_n(t)\,dt$ and $X(t) = \sum_{n=1}^N I_n(t)$. Therefore,

$$\sum_{n=1}^N W_n(t) = \sum_{n=1}^N \int_0^{S_1} I_n(t)\,dt = \int_0^{S_1} \sum_{n=1}^N I_n(t)\,dt = \int_0^{S_1} X(t)\,dt.$$

Now it follows from (3) that

$$\begin{aligned} L &= \mathbb{E}\left(\sum_{n=1}^N W_n(t)\right) \Big/ \mathbb{E}[S_1] \\ &= \frac{\mathbb{E}[N]}{\mathbb{E}[S_1]} \times \frac{\mathbb{E}\left(\sum_{n=1}^N W_n(t)\right)}{\mathbb{E}[N]} \\ &= \lambda W. \end{aligned}$$

4. Suppose that many gas particles carry out independent random walks in a container. Particles which hit the walls of the container become stuck, and we are interested in the distribution of the remaining freely-moving particles. This leads us to analyze the following model.

   Let $\{X(t), t \geq 0\}$ be a standard Brownian motion, with $X(0) = 0$. Define $A_t = \{-1 < X(s) < 1 \text{ for all } 0 \leq s \leq t\}$, the event that $X(s)$ has remained inside the unit disc for $0 \leq s \leq t$. Find the limiting density of $X(t)$ conditional on A_t as $t \to \infty$.

   Hint: it may be helpful to set up and solve an appropriate differential equation. Your argument does not need to be fully rigorous, but you should comment on unchecked assumptions.

   Solution: We look for a density function $f(x)$ for $0 \leq x \leq 1$ such that if $Y(0) \sim f$ and $Y(t) - Y(0)$ is a standard Brownian motion independent of $Y(0)$ then $Y(t)$ conditional on $A_t^Y = \{-1 < Y(s) < 1 \text{ for all } 0 \leq s \leq t\}$ has distribution f for all t. Supposing that the theory for countable-state Markov chains applies in this uncountable-state setting, the density of $X(t)$ conditional on A_t should converge to this unique stationary distribution.

   $Y(\delta)$ has density $f(x) - \delta \frac{d^2 f}{dx^2} + o(\delta)$. Thus, $Y(\delta)$ given A_δ^Y has density

$$\left(f(x) - \delta \frac{d^2 f}{dx^2} + o(\delta) \right) \Big(1 - c\delta + o(\delta) \Big)^{-1} \tag{4}$$

   for $-1 < x < 1$, where $c = \frac{d}{dt}(1 - \mathbb{P}[A_t^Y])$, assuming this derivative exists. For the numerator of (4) we have not directly considered trajectories which leave $[-1, 1]$ and return to a neighborhood of x since these have probability $o(\delta)$. By hypothesis, (4) equals $f(x)$ and taking $\delta \to 0$ then gives

$$\frac{d^2 f}{dx^2} = -cf, \tag{5}$$

   with boundary conditions $f(-1) = f(1) = 0$. The probability density solving this is $f(x) = \frac{\pi}{4}\cos(\pi x/2)$.