Applied Probability Qualifying Review Exam Questions
May, 2010
Time allowed: approximately 1/2 hr per question

1. Let $\{X_n, n \geq 0\}$ be a Markov chain taking values in $\{1, 2, \ldots, m\}$ and having one-step transition probabilities $P_{ij} = \mathbb{P}[X_n = j \mid X_{n-1} = i]$. Let $\mathcal{A}$ be a subset of the statespace, $\mathcal{A} \subset \{1, 2, \ldots, m\}$, and define $N = \min\{n : X_n \in \mathcal{A}\}$. Find a set of equations whose solution gives the following:

   (i) The mean time until $\{X_n\}$ hits $\mathcal{A}$ if the chain is started at i, i.e., $\mathbb{E}[N \mid X_0 = i]$.

   (ii) The probability that $\{X_n\}$ first hits $\mathcal{A}$ at j, i.e., $\mathbb{P}[X_N = j \mid X_0 = i]$.

2. This question considers a model for the fraction X_n of individuals in the nth generation of a certain population having a particular genetic trait (say, a gene for male-pattern baldness). Thinking of the population as being large, we suppose X_n is a continuous-valued random variable taking values in $[0, 1]$. In addition, we suppose that $\{X_n, n \geq 0\}$ has the Markov property. Further, we suppose that the genetic trait is neutral, meaning that it neither helps nor harms the individual, and so $\mathbb{E}[X_n \mid X_{n-1}] = X_{n-1}$. To allow for chance variation in transmission of the trait between generations, we suppose that $\mathrm{Var}[X_n \mid X_{n-1}] = cX_{n-1}(1 - X_{n-1})$ where c is some unknown constant. This can be thought of as generalizing the binomial distribution. Finally, suppose that $X_0 = p$ for some $0 < p < 1$.

   (i) Prove that X_n converges in some specified sense to either zero or one, i.e., in this model the trait will eventually die out or spread through the entire population.

   (ii) Find the probability that the trait eventually spreads through the entire population.

   (iii) Use a martingale stopping argument to find a lower bound on the probability that the fraction of the population with the trait never exceeds q for $q > p$.

3. Let $N(t)$ be a counting process describing customers arriving at a queue, with W_n being the time spent in the queue by the nth customer and $X(t)$ being the number of customers in the queue at time t. Let $\{S_n, n \geq 1\}$ be the sequence of times at which $X(t)$ jumps from zero to one. Suppose that these are regeneration times for the system, i.e., $\{X(t+u), t \geq 0\}$ is conditionally independent of $\{X(s), 0 \leq s \leq u\}$ given $\{S_n = u\}$ and has a conditional distribution which does not depend on n. In other words, the system resets when the queue is empty. Suppose that $\mathbb{E}[S_n - S_{n-1}] = \mu < \infty$. Beyond these assumptions, we allow consideration of an arbitrary queueing structure. Show that

   $$L = \lambda W$$

   where $\quad L = \lim_{t\to\infty} \frac{1}{t} \int_0^t X(u)\, du, \quad \lambda = \lim_{t\to\infty} N(t)/t, \quad \text{and} \quad W = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} W_k.$

4. Suppose that many gas particles carry out independent random walks in a container. Particles which hit the walls of the container become stuck, and we are interested in the distribution of the remaining freely-moving particles. This leads us to analyze the following model.

   Let $\{X(t), t \geq 0\}$ be a standard Brownian motion, with $X(0) = 0$. Define $A_t = \{-1 < X(s) < 1 \text{ for all } 0 \leq s \leq t\}$, the event that $X(s)$ has remained inside the unit disc for $0 \leq s \leq t$. Find the limiting density of $X(t)$ conditional on A_t as $t \to \infty$.

Hint: it may be helpful to set up and solve an appropriate differential equation. Your argument does not need to be fully rigorous, but you should comment on unchecked assumptions.