

Applied Probability Qualifying Review Exam Questions

May, 2010

Time allowed: approximately 1/2 hr per question

1. A village of $N + 1$ people suffers an epidemic of an infectious disease. Let $X(t)$ be the number of ill people at time t , with one initially infected individual so that $X(0) = 1$. Suppose $X(t)$ follows a continuous time Markov chain with transitions from n to $n + 1$ occurring at rate $\lambda n(N + 1 - n)$. Let T_N be the time at which everyone in the village has become sick. Find an expression for $\mathbb{E}[T_N]$ and show that $\mathbb{E}[T_N]$ is a decreasing function of N .

Solution: The expected time taken for $X(t)$ to jump from n to $n + 1$ is $[\lambda n(N + 1 - n)]^{-1}$ and so

$$\begin{aligned} \mathbb{E}[T_N] &= \frac{1}{\lambda} \sum_{n=1}^N \frac{1}{n(N + 1 - n)} = \frac{1}{\lambda(N + 1)} \sum_{n=1}^N \frac{1}{n} + \sum_{n=1}^N \frac{1}{N + 1 - n} \\ &= \frac{2}{\lambda(N + 1)} \sum_{n=1}^N \frac{1}{n}. \end{aligned}$$

To show this is decreasing in N , set $S_N = \sum_{n=1}^N \frac{1}{n}$ and write

$$\begin{aligned} \frac{\lambda}{2} (\mathbb{E}[T_N] - \mathbb{E}[T_{N+1}]) &= \frac{S_N}{N + 1} - \frac{S_{N+1}}{N + 2} \\ &= \frac{(N + 2)S_N - (N + 1)[S_N + 1/(N + 1)]}{(N + 1)(N + 2)} \\ &= \frac{S_N - 1}{(N + 1)(N + 2)} \geq 0 \quad \text{for } N \geq 1. \end{aligned}$$

2. We investigate a simple model for the spatial growth of a skin cancer. At each time t , there is a skin cell located at each point (m, n) of the integer lattice \mathbb{Z}^2 . Each cell's cancer status is either benign (B) or malignant (M). Each cell lives for an exponentially distributed amount of time, with rate λ_B or λ_M depending on its status, at which point it divides into two daughter cells each with the same cancer status as the parent. One daughter cell remains at the point of division and the other replaces one of the four neighboring cells, each chosen with probability $1/4$. The replaced cell leaves the system. Initially, there is a single M -cell at time $t = 0$.

(a) Let $X(t)$ be the number of M -cells at time t . Let X_n be the embedded discrete-time process consisting of the sequence of values taken by $X(t)$. Is $X(t)$ a Markov chain? Argue that X_n is a Markov chain and establish its transition probabilities.

(b) Find the chance that the cancer dies out as a function of the 'carcinogenic advantage' $\kappa = \lambda_M/\lambda_B$, supposing that $\kappa > 1$.

Solution:

(a) Note that $X(t)$ changes only when a cell divides and replaces a neighboring cell with a different cancer status. Let $N(t)$ be the number of (unordered) pairs of neighboring locations with different cancer status at time t . Then, $X(t)$ increases by 1 at rate $\lambda_m N(t)/4$ and $X(t)$

decreases by 1 at rate $\lambda_b N(t)/4$. Since $N(t)$ is not part of the state of the process $X(t)$ there is no reason to believe that $X(t)$ is a Markov chain.

However, the probability that $X(t)$ increases by 1 at its next transition is

$$(\lambda_m N(t)/4)/(\lambda_b N(t)/4 + \lambda_m N(t)/4) = \lambda_m/(\lambda_b + \lambda_m),$$

which does not depend on the state of the process. The other possibility is that $X(t)$ decreases by 1 at its next transition, with probability $\lambda_b/(\lambda_b + \lambda_m)$. Since this is true regardless of the history of $X(t)$, we find that X_n is a Markov chain with these transition probabilities.

(b) Let $p_n = \mathbb{P}[\text{Cancer extinct from } n]$ and $\pi = \lambda_m/(\lambda_b + \lambda_m)$ we condition to get the recursion

$$p_n = \pi p_{n+1} + (1 - \pi)p_{n-1}.$$

This difference equation has general solution $p_n = A + B\kappa^{-n}$. Since $p_0 = 1$ and $p_\infty = 0$, we infer $p_n = \kappa^{-n}$ and so $p_1 = \kappa^{-1}$.

3. A company has a quantity $Y(t)$ of a certain product at time t . Whenever the supply of this produce is entirely depleted, an order of size q is placed from an outside source. Each order has a cost $C + Dq^\alpha$ for $\alpha \geq 1$. The order is supposed to arrive instantaneously, so $Y(t)$ jumps from 0 to q . Between orders, the inventory behaves like a Brownian motion with drift $\mu < 0$ and infinitesimal variance σ^2 . Find the value of q that minimizes the long run expected cost per unit time.

Solution:

Let $X(t)$ be a Brownian motion with drift $\mu < 0$ and infinitesimal variance σ^2 having $X(0) = q$. $Z(t) = X(t) - \mu t$ is a martingale. $T = \inf\{t : X(t) = 0\}$ is a stopping time. We apply the Martingale stopping theorem, checking that $\mathbb{E}[T] < \infty$ and $\mathbb{E}[|Z(t+s) - Z(s)|]$ is bounded for each s . In the context of the course, we are assuming that the stopping theorem we proved for discrete-time martingales extends to the continuous-time case. Then, $\mathbb{E}[Z(T)] = q$ and so $\mathbb{E}[T] = -q/\mu$.

The cost of the system is a renewal/reward process and so the long run cost per unit time is $(-\mu/q)(C + Dq^\alpha)$ which is minimized by choosing $q = \left[\frac{C}{D(\alpha-1)}\right]^{1/\alpha}$. This does not depend on μ or σ .

4. Let $N_1(t)$ be a Poisson process with rate λ . Let $N_2(t)$ be a non-lattice renewal process, independent of $N_1(t)$, whose interarrival times have finite mean. Define $N(t) = N_1(t) + N_2(t)$ and suppose that $N(t)$ is a renewal process. Write F for the interarrival distribution of $N(t)$, with \bar{F} denoting the complementary c.d.f., μ denoting the mean and S_n denoting the corresponding sequence of arrival times.

(a) Let $Y(t)$ be the residual life process for $N(t)$, i.e. $Y(t) = S_{N(t)+1} - t$. Show that

$$\lim_{t \rightarrow \infty} \mathbb{P}(Y(t) \leq x) = \int_0^x \frac{1}{\mu} \bar{F}(y) dy.$$

(b) Let the interarrival times for $N_2(t)$ have distribution F_2 , complementary c.d.f. \bar{F}_2 , and mean μ_2 . Show that

$$\int_x^\infty \frac{1}{\mu} \bar{F}(y) dy = e^{-\lambda x} \int_x^\infty \frac{1}{\mu_2} \bar{F}_2(y) dy.$$

(c) Show that $\bar{F}_2(x) = \exp\{-cx\}$ for a value of c to be determined.

[Note: it follows fairly immediately from this calculation that $N(t)$ is a renewal process if and only if $N_2(t)$ is a Poisson process.]

Solution:

(a) Conditioning on $S_{N(t)}$ gives

$$\mathbb{P}(Y(t) \leq x) = \int_{t-x}^t dF_{S_{N(t)}} = \int_{t-x}^t \bar{F}(t-s) dm(s).$$

Applying the key renewal theorem (check the conditions!) gives the required expression.

(b) Note that $\{Y(t) > x\} = \{Y_1(t) > x\} \cap \{Y_2(t) > x\}$ and so

$$\mathbb{P}(Y(t) > x) = e^{-\lambda x} \mathbb{P}(Y_2(t) > x).$$

Now apply part (a) to both sides.

(c) Considering the first arrival time, we get $\bar{F}(x) = e^{-\lambda x} \bar{F}_2(x)$. Applying this to the result of part(ii) gives

$$\int_x^\infty \frac{1}{\mu} e^{-\lambda y} \bar{F}_2(y) dy = \int_x^\infty \frac{1}{\mu_2} e^{-\lambda x} \bar{F}_2(y) dy.$$

Differentiation gives

$$\frac{1}{\mu} e^{-\lambda x} \bar{F}_2(x) = \lambda e^{-\lambda x} \int_x^\infty \frac{1}{\mu_2} \bar{F}_2(y) dy + \frac{e^{-\lambda x}}{\mu} \bar{F}_2(x).$$

This simplifies to

$$\bar{F}_2(x) = \frac{\lambda\mu}{\mu_2 - \mu} \int_x^\infty \bar{F}_2(s)$$

which has solution $\bar{F}_2(x) = e^{-cx}$ for $c = \lambda\mu/(\mu_2 - \mu)$.