Applied Probability Qualifying Review Exam Questions
May, 2010
Time allowed: approximately 1/2 hr per question

1. A village of $N+1$ people suffers an epidemic of an infectious disease. Let $X(t)$ be the number of ill people at time t, with one initially infected individual so that $X(0) = 1$. Suppose $X(t)$ follows a continuous time Markov chain with transitions from n to $n+1$ occurring at rate $\lambda n(N+1-n)$. Let T_N be the time at which everyone in the village has become sick. Find an expression for $\mathbb{E}[T_N]$ and show that $\mathbb{E}[T_N]$ is a decreasing function of N.

2. We investigate a simple model for the spatial growth of a skin cancer. At each time t, there is a skin cell located at each point (m, n) of the integer lattice $\mathbb{Z}^2$. Each cell's cancer status is either benign (B) or malignant (M). Each cell lives for an exponentially distributed amount of time, with rate λ_B or λ_M depending on its status, at which point it divides into two daugher cells each with the same cancer status as the parent. One daughter cell remains at the point of division and the other replaces one of the four neighboring cells, each chosen with probability $1/4$. The replaced cell leaves the system. Initially, there is a single M-cell at time $t = 0$.

   (a) Let $X(t)$ be the number of M-cells at time t. Let X_n be the embedded discrete-time process consisting of the sequence of values taken by $X(t)$. Is $X(t)$ a Markov chain? Argue that X_n is a Markov chain and establish its transition probabilities.

   (b) Find the chance that the cancer dies out as a function of the 'carcinogenic advantage' $\kappa = \lambda_M/\lambda_B$, supposing that $\kappa > 1$.

3. A company has a quantity $Y(t)$ of a certain product at time t. Whenever the supply of this produce is entirely depleted, an order of size q is placed from an outside source. Each order has a cost $C + Dq^\alpha$ for $\alpha \geq 1$. The order is supposed to arrive instantaneously, so $Y(t)$ jumps from 0 to q. Between orders, the inventory behaves like a Brownian motion with drift $\mu < 0$ and infinitesimal variance σ^2. Find the value of q that minimizes the long run expected cost per unit time.

4. Let $N_1(t)$ be a Poisson process with rate λ. Let $N_2(t)$ be a non-lattice renewal process, independent of $N_1(t)$, whose interarrival times have finite mean. Define $N(t) = N_1(t) + N_2(t)$ and suppose that $N(t)$ is a renewal process. Write F for the interarrival distribution of $N(t)$, with $\overline{F}$ denoting the complementary c.d.f., μ denoting the mean and S_n denoting the corresponding sequence of arrival times.

   (a) Let $Y(t)$ be the residual life process for $N(t)$, i.e. $Y(t) = S_{N(t)+1} - t$. Show that

$$\lim_{t\to\infty} \mathbb{P}(Y(t) \leq x) = \int_0^x \frac{1}{\mu}\overline{F}(y)\, dy.$$

   (b) Let the interarrival times for $N_2(t)$ have distribution F_2, complementary c.d.f. $\overline{F}_2$, and mean μ_2. Show that

$$\int_x^\infty \frac{1}{\mu}\overline{F}(y)\, dy = e^{-\lambda x} \int_x^\infty \frac{1}{\mu_2}\overline{F}_2(y)\, dy.$$

(c) Show that $\overline{F}_2(x) = \exp\{-cx\}$ for a value of c to be determined.

[Note: it follows fairly immediately from this calculation that $N(t)$ is a renewal process if and only if $N_2(t)$ is a Poisson process.]