Applied Probability Qualifying Review Exam Questions
Tuesday May 29, 2012
9a.m. to 1p.m.

1. The number of individuals in a population maintained by immigration and depleted by individual deaths and collective disasters is modeled by a continuous time Markov chain, $X(t)$. New individuals enter the population at rate λ. Each individual in the population dies, independently, at rate μ. In addition, at rate δ the whole population is eliminated. Thus, the nonzero transition rates are

$$\begin{aligned} q_{i,i+1} &= \lambda \\ q_{i,i-1} &= \mu i \quad \text{for } i \geq 2 \\ q_{i,0} &= \delta \quad \text{for } i \geq 2 \\ q_{1,0} &= \mu + \delta. \end{aligned}$$

(a) Is $X(t)$ reversible?

(b) Find the limiting mean of $X(t)$ as t increases.

[Hint: direct computation via evaluating the stationary distribution is both tricky and unnecessary. Two easier ways to proceed are (i) set up and solve a differential equation satisfied by $\mathbb{E}[X(t)]$, or (ii) consider the marginal distribution of the time that each immigrant remains in the population.]

(c) Show that $X(t)$ has a unique stationary distribution (you do not need to obtain it explicitly).

Solution: (a) No. For example, the rate of going directly from i to zero is positive for all i, whereas the rate of going directly from zero to i is zero for $i > 1$. Formally, let $\pi = (\pi_0, \pi_1, \dots)$ be a stationary distribution (supposing it exists). Since the state space consists of a single communicating class, one must have $\pi_i > 0$ for all i. Then, reversibility implies that $\pi_0 q_{0,2} = \pi_2 q_{2,0}$ which is a contradiction.

(b) [method (i)] From the transition probabilities, we get

$$\mathbb{E}[X(t+h) \mid X(t)] = X(t) + \lambda h - (\mu + \delta) X(t) h + o(h).$$

Assuming that the expectation of the remainder is $o(h)$, one has

$$\frac{d}{dt}\mathbb{E}[X(t)] = \lambda - (\mu + \delta)\mathbb{E}[X(t)].$$

As $t \to \infty$ this converges to the fixed point at

$$\lim_{t \to \infty} \mathbb{E}[X(t)] = \frac{\lambda}{\mu + \delta}.$$

[method (ii)] Each immigrant has a marginal death rate of $\mu + \delta$, and thus an expected time $(\mu + \delta)^{-1}$ in the population. For large t, the expected number of immigrant arriving by time t is λt. Therefore, the limiting mean number of immigrants present in the interval $[0, t]$ is

$\lambda/(\mu+\delta)^{-1}$. This is a non-lattice regenerative process, and so (by the regenerative process limit theorem) the limiting time average is equal to the limiting expectation.

(c) As we have already noted, the state space forms a single communicating class. The chain is acyclic, and it therefore has a unique stationary distribution if it is positive recurrent. The state 0 is clearly positive recurrent, and so the whole communicating class is positive recurrent.

2. For a general renewal process $N(t)$, we study the second moment function $V(t) = \mathbb{E}[N(t)^2]$. We use standard renewal theory notation, so that $m(t) = \mathbb{E}[N(t)]$ is the renewal function, and X_n is the nth arrival time with cumulative distribution function F_n.

   (a) Show that $V(t) = \sum_{n=1}^{\infty}(2n-1)\mathbb{P}\{N(t) \geq n\}$.

   (b) Hence, or otherwise, show that $V(t) = m(t) + 2\int_0^t m(t-s)\,dm(s)$.

   [Hint: there are at least two ways to approach this: (i) by direct calculation, using the identity $m(t) = \sum_{n=1}^{\infty} F_n(t)$ together with part (a); (ii) via Laplace transforms.]

   (c) Check that the formula in (b) gives the correct result when $N(t)$ is a Poisson process.

   Solution: Now, for the left hand side,

$$\begin{aligned} V(t) &= \sum_n (2n-1)\mathbb{P}\{N(t) \geq n\} \\ &= \sum_n (2n-1)\mathbb{P}\{S_n \leq t\} \\ &= \sum_n (2n-1)F_n(t) \end{aligned} \tag{1}$$

   (b) Use the identity

$$m(t) = \sum_{n=1}^{\infty} F_n(t) \tag{2}$$

   to calculate

$$\begin{aligned} \int_0^t m(t-s)\,dm(s) &= \int_0^t \sum_{m=1}^{\infty} F_m(t-s) \sum_{n=1}^{\infty} dF_n(s) \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \int_0^t F_m(t-s)dF_n(s) \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{m+n}(t) \\ &= \sum_{k=1}^{\infty}(k-1)F_k(t). \end{aligned} \tag{3}$$

   Putting together (3) and (1) gives the solution to (b). For a Poisson process with rate λ, we know that $m(t) = \lambda t$ and so (b) gives

$$V(t) = \lambda t + 2\lambda^2 \int_0^t (t-s)\,ds \quad = \quad \lambda t + \lambda^2 t^2,$$

   which checks (c).

3. Let $\{Y_n, n = 0, 1, \dots\}$ be a martingale with $\mathbb{E}[Y_n] = 0$ and $\mathbb{E}[Y_n^2] < \infty$ for all n. Show that, for $x > 0$,

$$\mathbb{P}\left\{\max_{1\le k\le n} Y_k > x\right\} \le \frac{\mathbb{E}[Y_n^2]}{\mathbb{E}[Y_n^2] + x^2}.$$

Solution: For $x > 0$ and $c > 0$, note that

$$\mathbb{P}\left\{\max_{1\le k\le n} Y_k > x\right\} \le \mathbb{P}\left\{\max_{1\le k\le n} (Y_k + c)^2 > (x + c)^2\right\}.$$

Since $(Y_k + c)^2$ is a convex function of Y_k, it defines a submartingale. Applying Kolmogorov's submartingale inequality, we get

$$\mathbb{P}\left\{\max_{1\le k\le n} (Y_k + c)^2 > (x + c)^2\right\} \le \mathbb{E}\left[\frac{(Y_n + c)^2}{(x + c)^2}\right].$$

Now, putting $c = \mathbb{E}[Y_n^2]/x$ gives the required result.

4. Let $X(t)$ and $Y(t)$ be two independent Brownian motions, both having infinitesimal variance parameter σ^2. Thinking of $\big(X(t), Y(t)\big)$ as Brownian motion on the plane, we investigate the transformation into polar coordinates. Thus, we define $X(t) = R(t)\cos\Theta(t)$ and $Y(t) = R(t)\sin\Theta(t)$, which can also be written as $R(t) = \sqrt{X(t)^2 + Y(t)^2}$ and $\Theta(t) = \arctan\big(Y(t)/X(t)\big)$.

(a) Compute the infinitesimal conditional mean and variance given by

$$\text{(i)}\quad \lim_{h\to 0}\frac{1}{h}\mathbb{E}[R(t+h) - R(t)\,|\,R(t), \Theta(t)]; \qquad \text{(ii)}\quad \lim_{h\to 0}\frac{1}{h}\mathrm{Var}[R(t+h) - R(t)\,|\,R(t), \Theta(t)].$$

This essentially amounts to a bivariate application of Ito's formula. Since only the univariate case was covered in class, you should not just state and use a multivariate result. You should carry out a Taylor series approximation, discussing which terms become negligible in the limit as $h \to 0$ but without supplying a formal proof of this. It may be helpful to notice that conditioning on $\big(R(t), \Theta(t)\big)$ is equivalent to conditioning on $\big(X(t), Y(t)\big)$.

(b) Is the vector stochastic process $\{\big(R(t), \Theta(t)\big), t \ge 0\}$ a diffusion process? Which, if any, of the scalar processes $\{R(t), t \ge 0\}$ and $\{\Theta(t), t \ge 0\}$ are diffusion processes? Explain.

Solution: (a) Write a Taylor series expansion

$$\begin{aligned} R(t+h) &= R(t) + \frac{\partial R}{\partial X}[X(t+h) - X(t)] + \frac{\partial R}{\partial Y}[Y(t+h) - Y(t)] + (1/2)\frac{\partial^2 R}{\partial X^2}[X(t+h) - X(t)]^2 \\ &\quad + (1/2)\frac{\partial^2 R}{\partial Y^2}[Y(t+h) - Y(t)]^2 + \frac{\partial^2 R}{\partial X \partial Y}[X(t+h) - X(t)][Y(t+h) - Y(t)] + A \quad (4) \end{aligned}$$

Assume that $\mathbb{E}[A] = o(h)$, noting that Ito's formula for univariate diffusions tells us that this is true in the univariate case. Now compute

$$\frac{\partial R}{\partial X} = \frac{X}{R}; \quad \frac{\partial R}{\partial Y} = \frac{Y}{R}; \quad \frac{\partial^2 R}{\partial X^2} = \frac{1}{R} - \frac{X^2}{R^3}; \quad \frac{\partial^2 R}{\partial Y^2} = \frac{1}{R} - \frac{Y^2}{R^3}; \quad \frac{\partial^2 R}{\partial X \partial Y} = -\frac{XY}{R^3}.$$

Taking expectations of (4) gives

$$\mathbb{E}[R(t+h)-R(t)\,|\,X(t),Y(t)] = \frac{1}{2}\left(\frac{1}{R(t)}-\frac{X(t)^2}{R(t)^3}\right)\sigma^2 h + \frac{1}{2}\left(\frac{1}{R(t)}-\frac{Y(t)^2}{R(t)^3}\right)\sigma^2 h + o(h),$$

and so

$$\lim_{h\to 0}\frac{1}{h}\mathbb{E}[R(t+h)-R(t)\,|\,R(t),\Theta(t)] = \left(\frac{2}{R(t)}-\frac{X(t)^2+Y(t)^2}{R(t)^3}\right)\sigma^2 \quad = \quad \frac{\sigma^2}{2R(t)}.$$

For the infinitesimal variance, we assume that the variance of the remainder from the first order Taylor series expansion of (4) is $o(h)$, to obtain

$$\mathrm{Var}[R(t+h)-R(t)\,|\,X(t),Y(t)] = \left(\frac{X(t)^2}{R(t)^2}+\frac{Y(t)^2}{R(t)^2}\right)\sigma^2 h + o(h) \quad = \quad \sigma^2 h + o(h).$$

(b) $\{\big(R(t),\Theta(t)\big), t\geq 0\}$ is a diffusion process since it is a continuous and invertible function of a diffusion process.

$\{R(t), t\geq 0\}$ has continuous sample paths, and from part (a) we discovered that it has the Markov property, since the infinitesimal mean and variance happen to depend only on $R(t)$ and not on $\Theta(t)$.

$\{\Theta(t), t\geq 0\}$ has continuous sample paths, but it does not have the Markov property. Heuristically, this can be reasoned by noting that stochastic fluctuations in $\Theta(t)$ will be much greater when $\big(X(t),Y(t)\big)$ is close to the origin. Thus a recent history of rapid fluctuations will predict further intense fluctuations in the near future. This contradicts the Markov property. Formally, one can confirm this by carrying out the corresponding computation of infinitesimal moments in (a) for $\Theta(t)$.