Applied Probability Qualifying Review Exam Questions
Tuesday May 29, 2012
9a.m. to 1p.m.

1. The number of individuals in a population maintained by immigration and depleted by individual deaths and collective disasters is modeled by a continuous time Markov chain, $X(t)$. New individuals enter the population at rate λ. Each individual in the population dies, independently, at rate μ. In addition, at rate δ the whole population is eliminated. Thus, the nonzero transition rates are

$$\begin{aligned} q_{i,i+1} &= \lambda \\ q_{i,i-1} &= \mu i \quad \text{for } i \geq 2 \\ q_{i,0} &= \delta \quad \text{for } i \geq 2 \\ q_{1,0} &= \mu + \delta. \end{aligned}$$

   (a) Is $X(t)$ reversible?

   (b) Find the limiting mean of $X(t)$ as t increases.

   [Hint: direct computation via evaluating the stationary distribution is both tricky and unnecessary. Two easier ways to proceed are (i) set up and solve a differential equation satisfied by $\mathbb{E}[X(t)]$, or (ii) consider the marginal distribution of the time that each immigrant remains in the population.]

   (c) Show that $X(t)$ has a unique stationary distribution (you do not need to obtain it explicitly).

2. For a general renewal process $N(t)$, we study the second moment function $V(t) = \mathbb{E}[N(t)^2]$. We use standard renewal theory notation, so that $m(t) = \mathbb{E}[N(t)]$ is the renewal function, and X_n is the nth arrival time with cumulative distribution function F_n.

   (a) Show that $V(t) = \sum_{n=1}^{\infty}(2n-1)\mathbb{P}\{N(t) \geq n\}$.

   (b) Hence, or otherwise, show that $V(t) = m(t) + 2\int_0^t m(t-s)\, dm(s)$.

   [Hint: there are at least two ways to approach this: (i) by direct calculation, using the identity $m(t) = \sum_{n=1}^{\infty} F_n(t)$ together with part (a); (ii) via Laplace transforms.]

   (c) Check that the formula in (b) gives the correct result when $N(t)$ is a Poisson process.

3. Let $\{Y_n, n = 0, 1, \dots\}$ be a martingale with $\mathbb{E}[Y_n] = 0$ and $\mathbb{E}[Y_n^2] < \infty$ for all n. Show that, for $x > 0$,

$$\mathbb{P}\left\{\max_{1 \leq k \leq n} Y_k > x\right\} \leq \frac{\mathbb{E}[Y_n^2]}{\mathbb{E}[Y_n^2] + x^2}.$$

4. Let $X(t)$ and $Y(t)$ be two independent Brownian motions, both having infinitesimal variance parameter σ^2. Thinking of $\big(X(t), Y(t)\big)$ as Brownian motion on the plane, we investigate the transformation into polar coordinates. Thus, we define $X(t) = R(t)\cos\Theta(t)$ and $Y(t) = R(t)\sin\Theta(t)$, which can also be written as $R(t) = \sqrt{X(t)^2 + Y(t)^2}$ and $\Theta(t) = \arctan\big(Y(t)/X(t)\big)$.

(a) Compute the infinitesimal conditional mean and variance given by

$$\text{(i)} \quad \lim_{h\to 0} \frac{1}{h}\mathbb{E}[R(t+h) - R(t) \mid R(t), \Theta(t)]; \qquad \text{(ii)} \quad \lim_{h\to 0} \frac{1}{h}\text{Var}[R(t+h) - R(t) \mid R(t), \Theta(t)].$$

This essentially amounts to a bivariate application of Ito's formula. Since only the univariate case was covered in class, you should not just state and use a multivariate result. You should carry out a Taylor series approximation, discussing which terms become negligible in the limit as $h \to 0$ but without supplying a formal proof of this. It may be helpful to notice that conditioning on $\big(R(t), \Theta(t)\big)$ is equivalent to conditioning on $\big(X(t), Y(t)\big)$.

(b) Is the vector stochastic process $\{\big(R(t), \Theta(t)\big), t \geq 0\}$ a diffusion process? Which, if any, of the scalar processes $\{R(t), t \geq 0\}$ and $\{\Theta(t), t \geq 0\}$ are diffusion processes? Explain.